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1990 J. Phys. A: Math. Gen. 23 5153

(http://iopscience.iop.org/0305-4470/23/22/010)

J. Phys. A: Math. Gen. 23 (1990) 5153-5168. Printed in the UK

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Received 18 December 1989

Abstract. The lift of a closed 2-form Ω on a manifold Q to a closed 2-form on TQ may be achieved by pulling back the canonical symplectic structure on T^*Q by the bundle map Ω: $TQ \to T^*Q$. It is also known how to lift a Poisson structure on Q to a Poisson structure on TQ. We call these lifted structures 'tangent' structures. The notion of a Dirac structure is reviewed. This is a hybrid of Poisson and pre-symplectic structures, which may be thought of as a (singular) foliation of Q by pre-symplectic leaves. The main result of this paper is a single method which achieves the lift to TQ of either a Poisson or a pre-symplectic structure on Q. Natural involution on TTQ is shown to preserve the lift to TTQ of the lifted structure on TQ. The method is then applied to Dirac structures, with the result that a Dirac structure on Q has a tangent lift to TQ generalizing the lifts of Poisson and pre-symplectic structures.

1. Introduction

The underlying structure in the study of Hamiltonian systems on a manifold Q is a Poisson algebra on some subalgebra of $C^\infty(Q)$, i.e. a Lie algebra bracket $\{\,,\}$ with an additional condition, called the Leibniz rule:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$

Thus a function f in the Poisson algebra gives rise to $\mathrm{ad}_f = \{f, \ \}$ which is a derivation on the Poisson algebra, and hence may be thought of as defining a vector field on Q.

Two natural ways for Poisson algebras to arise are through Poisson or pre-symplectic structures on Q. These structures are 2-tensors satisfying certain integrability conditions (see Weinstein 1983, or Abraham and Marsden 1978). A Poisson structure is defined as a bundle map $B: T^*Q \to TQ$ for which the Schouten bracket $[B, B]$ of the corresponding covariant 2-tensor with itself vanishes. The image of B is an integrable singular distribution, giving a foliation of Q whose leaves are symplectic (see Weinstein 1983). The Poisson algebra arises as follows: for any function $f \in C^\infty(Q)$ we define $B\,\mathrm{d}f = \mathrm{ad}_f$ so that $\{f, g\} = \langle B |\, \mathrm{d}f \wedge \mathrm{d}g\rangle$. With this notation the Schouten bracket $[B, B]$ is determined by

$$\tfrac{1}{2}\langle [B, B] |\, \mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h\rangle = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$$

so that $[B, B] = 0$ if and only if $\{\,,\}$ is a lie bracket; the Leibniz rule holds automatically since B takes values in TQ.

A pre-symplectic structure on Q is a skew-symmetric bundle map $\Omega: TQ \to T^*Q$. The integrability condition is that the exterior derivative $\mathrm{d}\Omega$ vanishes. In this case the vector fields ad_f are defined as follows: if $X \lrcorner \Omega = \mathrm{d}f$ for some function f, we write $X = X_f = \mathrm{ad}_f$. Then $\{f, g\} = \Omega(X_g, X_f)$; note that X_f is only defined up to a vector field in $\ker \Omega$, i.e. a gauge field on Q (see Hanson *et al* 1976). However, $\{f, g\}$ is well defined on the subset of $C^\infty(Q)$ which is constant along $\ker \Omega$. The condition

$$\mathrm{d}\Omega(X_f, X_g, X_h) = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$$

shows that $\mathrm{d}\Omega = 0$ if and only if $\{\,,\}$ is a lie bracket.

0305-4470/90/225153+16$03.50

A natural extension of these two examples is the Dirac structure (Courant 1989), a sort of hybrid structure which may be thought of as a foliation of Q whose leaves are pre-symplectic. Dirac structures are defined as subbundles $L \subseteq TQ \oplus T^*Q$, which are maximally isotropic under the natural pairing $\langle\ ,\ \rangle_+$ on $TQ \oplus T^*Q$ given by $\langle (X, \omega), Y, \mu)\rangle_+ = \omega(Y) + \mu(X)$; this is a natural generalization of a skew-symmetric bundle map between TQ and T^*Q. The integrability condition for Dirac structures is given by the vanishing of a certain 3-tensor on the vector bundle L. If L is the graph of a bundle map, we recover the Poisson or the pre-symplectic case (see Courant and Weinstein 1988, Courant 1989). These hybrid structures are named for Dirac because they form the natural setting for applying Dirac's theory of constraints, a method for passing Poisson brackets to submanifolds (see Gotay *et al* 1978, Sniatycki 1974, Dirac 1964).

The infinite-dimensional case of Dirac structures has been applied to the study of local Hamiltonian and local symplectic operators, with applications to the theory of Hamiltonian pairs and complete integrability, see Dorfman (1987).

Given a symplectic structure on a manifold Q, there is a natural way to 'lift' the structure to a symplectic structure on the bundle TQ. The symplectic structure on Q may be viewed as a bundle map $TQ \to T^*Q$, and thus it may be used to pull back the canonical symplectic structure Ω_Q on T^*Q; the result is a closed non-degenerate 2-form on TQ, i.e. a symplectic structure.

A similar situation exists in the Poisson case: a Poisson structure on a manifold Q has a naturally induced lift to a Poisson structure on TQ. This tangent lift has many properties, among them: it is natural with respect to tangents of Poisson maps (so that the tangent of a Poisson map $Q \to P$ is again a Poisson map $TQ \to TP$), a submanifold L of Q is Lagrangian if and only if TL is Lagrangian in TQ, and a vector field X on Q is Hamiltonian if and only if its graph $\Gamma(X) \subseteq TQ$ is Lagrangian. The tangent Poisson structure has applications in control theory; see Alvarez-Sanchez (1986).

In this paper a method is given which works for establishing the lift to TQ of either a Poisson structure or a pre-symplectic structure on Q. This involves applying the natural involution to the bundle TTQ, and using a certain diffeomorphism $TT^*Q \to T^*TQ$ (see Tulczyjew 1977). As a corollary, we obtain the result that in both the Poisson and pre-symplectic cases the natural involution map $TTQ \to TTQ$ preserves the tangent lift to TTQ of the tangent structure on TQ. Our method is then applied to obtain the lift of a Dirac structure from Q to TQ. This construction generalizes the lifting of Poisson and pre-symplectic structures.

2. Poisson structures

A Poisson structure on a manifold Q is defined as a skew-symmetric bundle map π: $T^*Q \to TQ$, determined by a bivector field, so that $\pi = \pi^{ij}(\partial/\partial q^i) \wedge (\partial/\partial q^j)$, with the additional condition that the Schouten bracket of π with itself vanishes, i.e. $[\pi, \pi] = 0$.

A Poisson algebra on Q is a bracket $\{\ ,\ \}$ on a subset of $C^\infty(Q)$ such that $\{f, gh\} = g\{f, h\} + \{f, g\}h$ and $\{\ ,\ \}$ is a Lie algebra bracket. From these definitions we have the following theorem.

Theorem. A Poisson structure on Q determines a Poisson algebra on $C^\infty(Q)$ given by

$$\langle \pi | \mathrm{d}f \wedge \mathrm{d}g \rangle = \{f, g\}.$$

Proof. We only need to show that $\{\,,\}$ makes $C^\infty(Q)$ into a Poisson algebra, i.e.

$$\begin{aligned}\{f, gh\} &= \langle \pi | \mathrm{d}f \wedge \mathrm{d}(gh)\rangle \\ &= \langle \pi | \mathrm{d}f \wedge (g\,\mathrm{d}h + h\,\mathrm{d}g)\rangle \\ &= g\{f, h\} + h\{f, g\}.\end{aligned}$$

Now $\langle [\pi, \pi] | \mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h\rangle = 2(\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\})$ so $[\pi, \pi] = 0$ if and only if $\{\,,\}$ is a Lie algebra bracket. □

Example. A symplectic structure on Q is a non-degenerate closed 2-form Ω on Q, which we may consider as an invertible skew symmetric bundle map $\Omega: TQ \to T^*Q$. Thus its inverse is a skew symmetric bundle map $\Omega^{-1}: T^*Q \to TQ$ which we may interpret as a bivector-field π_Q. Then the relation $\langle X_f \wedge X_g \wedge X_h | \mathrm{d}\Omega\rangle = \langle [\pi_Q, \pi_Q] | \mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h\rangle$ (see Tulczyjew 1974) shows that this inverse bundle map is indeed a Poisson structure on Q.

Example (Lie-Poisson structure). Let $\mathfrak{g}$ be any Lie algebra. A Poisson algebra on $C^\infty(\mathfrak{g}^*)$ may be defined as follows: let $f, g \in C^\infty(\mathfrak{g}^*)$; then the Frechét derivatives of f and g at μ are maps $Df(\mu)$, $Dg(\mu): T_\mu \mathfrak{g}^* \to \mathbb{R}$, or maps $Df(\mu)$, $Dg(\mu): \mathfrak{g}^* \to \mathbb{R}$; since these maps are linear functionals on $\mathfrak{g}^*$, we may identify them with elements $\delta f/\delta\mu$, $\delta g/\delta\mu \in \mathfrak{g}$. Set

$$\{f, g\}(\mu) = \left\langle \mu \left| \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right]\right.\right\rangle.$$

This defines a Poisson structure on $\mathfrak{g}^*$; it follows that the Hamiltonian vector fields are given by $X_f(\mu) = \{f, \;\}(\mu) = ad^*_{\delta f/\delta\mu}\mu$. If we let $x^1, x^2, \ldots, x^n$ be a linear basis for $\mathfrak{g}$, we see that $\{x^i, x^j\} = \pi^{ij}_k x^k$ are the components of the Poisson bivector field for this Poisson structure, where the π^{ij}_k are the structure constants of the Lie algebra $\mathfrak{g}$.

Taking this one step further, we consider another example.

Example. Consider the semi-direct product Lie algebra $\mathfrak{g} \circledS \mathfrak{g}$ of $\mathfrak{g}$ with itself, whose bracket is given by $[(\mu, \nu), (\bar\mu, \bar\nu)] = ([\mu, \bar\mu], [\mu, \bar\nu] - [\bar\mu, \nu])$. Then the Lie-Poisson bracket is given by

$$\{f, g\}(\mu, \nu) = \left\langle \mu \left| \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right]\right.\right\rangle + \left\langle \nu \left| \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\nu}\right]\right.\right\rangle - \left\langle \nu \left| \left[\frac{\delta g}{\delta\mu}, \frac{\delta f}{\delta\nu}\right]\right.\right\rangle.$$

As above, we choose a basis $(\mu^i. \nu^j)$ of $\mathfrak{g} \circledS \mathfrak{g}$; then the Lie-Poisson structure is given by:

$$\{\nu^i, \nu^j\} = 0 \qquad \{\mu^i, \nu^j\} = \pi^{ij}_k \nu^k \qquad \{\mu^i, \mu^j\} = \pi^{ij}_k \mu^k.$$

(Note that we may write this as $\{\mu^i, \nu^j\} = (\partial\{\mu^i, \mu^j\}/\partial\mu^k)\nu^k$; we shall see this again later.)

3. Tangent symplectic and pre-symplectic structures

Consider a pre-symplectic structure on Q given by a skew symmetric bundle map $\Omega: TQ \to T^*Q$. We may use Ω to pull back to TQ the canonical symplectic structure

Ω_Q to T^*Q. We now carry this out in local coordinates; let (q^i, p_j) be local coordinates on T^*Q, and $(q^i, \dot{q}^j)$ local coordinates on TQ induced by a local chart (q^i) on Q. The pre-symplectic structure is given locally by $p_i = \Omega_{ij}\dot{q}^j$, where $\Omega_{ij} = \langle\Omega|(\partial/\partial q^i) \wedge (\partial/\partial q^j)\rangle$. Then the pullback $\Omega^*(\Omega_Q)$ is given by

$$\begin{aligned}
\mathrm{d}p_i \wedge \mathrm{d}q^i &= \mathrm{d}(\Omega_{ij}\dot{q}^j) \wedge \mathrm{d}q^i \\
&= \Omega_{ij,k}\dot{q}^j \,\mathrm{d}q^k \wedge \mathrm{d}q^i + \Omega_{ij}\,\mathrm{d}\dot{q}^j \wedge \mathrm{d}q^i \\
&= \sum_{i<j} (\Omega_{ij,k} - \Omega_{kj,i})\dot{q}^j \,\mathrm{d}q^k \wedge \mathrm{d}q^i + \Omega_{ij}\,\mathrm{d}\dot{q}^j \wedge \mathrm{d}q^i \\
&= \Omega_{ik,j}\dot{q}^j \,\mathrm{d}q^k \wedge \mathrm{d}q^i + \Omega_{ij}\,\mathrm{d}\dot{q}^i \wedge \mathrm{d}q^j \\
&= \Omega_{ij,k}\dot{q}^k \,\mathrm{d}q^j \wedge \mathrm{d}q^i + \Omega_{ij}\,\mathrm{d}\dot{q}^i \wedge \mathrm{d}q^j
\end{aligned}$$

where we have used $\mathrm{d}\Omega = 0$ in the form $\Omega_{ij,k} + \Omega_{ki,j} + \Omega_{jk,i} = 0$. Thus we get a pre-symplectic structure on TQ inherited from the structure Ω on Q. We call this the tangent lift to TQ of the 2-form on Q.

3.1. Natural involution

Recall that on the double tangent bundle of a manifold there is a map, the so-called natural involution, $\sim : TTQ \to TTQ$. Let (q^i) be a coordinate chart on Q, and let $(q^i, \dot{q}^j)$ be the induced coordinate chart on TQ; finally, about points $(q^i, \dot{q}^j)$ and $(q^i, \delta q^j)$ of TQ, let the induced charts on TTQ be denoted by $(q^i, \dot{q}^j, \delta q^i, \delta\dot{q}^j)$ and $(q^i, \delta q^j, \dot{q}^i, \delta\dot{q}^j)$, respectively (i.e. one applies dots or deltas to the first pair of coordinate functions). Then locally, natural involution is given by $\sim(q^i, \dot{q}^j, \delta q^i, \delta\dot{q}^j) = (q^i, \delta q^j, \dot{q}^i, \delta\dot{q}^j)$. Notice that $\sim^2 = \mathbf{1}_{TTQ}$. Although I have given only a local representation, $\sim$ is in fact a global map. See Tulczyjew (1977), or Abraham and Marsden (1978). Also note that it is not a bundle map.

3.2. Canonical involution

Now consider the manifolds TT^*Q and T^*TQ, endowed with the tangent symplectic structure and the canonical symplectic structure, respectively. Let (q^i) be a chart on Q and let $(q^i, p_j, \dot{q}^i, \dot{p}_j)$ and $(q^i, \dot{q}^j, p_i, \dot{p}_j)$ be the induced symplectic charts on TT^*Q and T^*TQ, respectively, so that the symplectic structures are given by $\mathrm{d}q^i \wedge \mathrm{d}\dot{p}_i + \mathrm{d}\dot{q}^i \wedge \mathrm{d}p_j$ and $\mathrm{d}q^i \wedge \mathrm{d}\dot{q}^i + \mathrm{d}p_i \wedge \mathrm{d}\dot{p}_i$, respectively. Canonical involution is the globally defined map $\alpha : T^*TQ \to TT^*Q$ that intertwines the given symplectic structures on these two manifolds; note that it is not really an involution, since α^2 is not defined. In local coordinates α is given by

$$\alpha(q^i, \dot{q}^j, p^r, \dot{p}^s) = (q^i, p^r, \dot{p}^s, \dot{q}^j).$$

Then clearly $\alpha^*(\mathrm{d}q^i \wedge \mathrm{d}\dot{p}_i + \mathrm{d}\dot{q}^i \wedge \mathrm{d}p_j) = \mathrm{d}q^i \wedge \mathrm{d}\dot{q}^i + \mathrm{d}p_i \wedge \mathrm{d}\dot{p}_i$. For a detailed discussion of the map α, see Tulczyjew (1977).

4. Tangent Poisson structures

Consider for the moment the bundle map $\Omega : TQ \to T^*Q$ of a symplectic structure on Q. We may take the tangent map $T\Omega : TTQ \to TT^*Q$ and apply the natural and canonical involution operators to get the following commutative diagram:

$$\begin{array}{ccc} TTQ & \longrightarrow & TT^*Q \\ \sim\downarrow & & \downarrow\alpha \\ TTQ & \longrightarrow & T^*TQ. \end{array}$$

Indeed a local calculation shows that the bottom map is exactly the 2-form $\Omega^*(\Omega_Q)$ on TQ that was previously computed.

Consider now the Poisson structure determined by a bundle map $B: T^*Q \to TQ$. Again we will take the tangent map $TB: TT^*Q \to TTQ$ and apply the canonical and natural involutions, resulting in the following diagram:

$$\begin{array}{ccc} TT^*Q & \longrightarrow & TTQ \\ \alpha\downarrow & & \downarrow\sim \\ T^*TQ & \longrightarrow & TTQ. \end{array}$$

Again a local calculation shows that the bottom map is a bundle map. In fact, it again defines a Poisson structure, this time on the manifold TQ.

Theorem. Let $(q^i, \dot{q}^j)$ be a local chart on TQ induced by (q^i) on Q, and let $\{q^i, q^j\} = \pi^{ij}$ be the Poisson brackets for B on Q. Then the bottom map of the diagram above determines a Poisson structure on TQ given locally by

$$\{q^i, q^j\} = 0 \qquad \{q^i, \dot{q}^j\} = \pi^{ij} \qquad \{\dot{q}^i, \dot{q}^j\} = \pi^{ij}{}_{,k}\dot{q}^k.$$

This is the Poisson structure first used by Alvarez-Sanchez in applications to control theory; see Alvarez-Sanchez (1986).

Proof. The proof is given by a calculation in local coordinates, keeping in mind that all of the maps above are globally defined.

The map $B: T^*Q \to TQ$ is a bundle map over the identity given in local coordinates (q^i) by $\dot{q}^i = \pi^{ji}p_j$. In the coordinates $(q, \dot{q})$ on TQ and (q, p) on T^*Q we compute:

$$TB|_{(q,p)}\left(\dot{q}^i \frac{\partial}{\partial q^i}\right) = \dot{q}^i \left(\frac{\partial q^j}{\partial q^i}\frac{\partial}{\partial q^j} + \frac{\partial p_r \pi^{rj}}{\partial q^i}\frac{\partial}{\partial \dot{q}^j}\right)$$

$$= \dot{q}^i \frac{\partial}{\partial q^i} + p_r \pi^{rj}{}_{,i}\dot{q}^i \frac{\partial}{\partial \dot{q}^j}$$

$$TB|_{(q,p)}\left(\dot{p}^i \frac{\partial}{\partial p_i}\right) = \dot{p}_i \left(\frac{\partial q^j}{\partial p_i}\frac{\partial}{\partial q^j} + \frac{\partial p_r \pi^{rj}}{\partial p_i}\frac{\partial}{\partial \dot{q}^j}\right)$$

$$= \dot{p}_i \frac{\partial p_r \pi^{rj}}{\partial p_i}\frac{\partial}{\partial \dot{q}^j}$$

$$= \dot{p}_i \pi^{ij}\frac{\partial}{\partial \dot{q}^j}$$

so that $TB\ (q, p, \dot{q}, \dot{p}) = (q, p_i\pi^{ij}, \dot{q}^j, \dot{p}_i\pi^{ij} + p_r\pi^{rj}{}_{,i}\dot{q}^i)$.

Consider now the map $\alpha: TT^*Q \to TTQ$; in local coordinates $(q, p, \dot{q}, \dot{p})$ on TT^*Q, α is given by $\alpha(q, p, \dot{q}, \dot{p}) = (q, \dot{q}, \dot{p}, p)$. Thus in terms of the induced coordinates

$(q, \dot{q}, p, \dot{p})$ we have the inverse map $\alpha^{-1}(q, \dot{q}, \dot{p}, p) = (q, \dot{p}, \dot{q}, p)$. Therefore the map $TB \circ \alpha^{-1}$ is given by

$$TB \circ \alpha^{-1}: (q, \dot{q}, p, \dot{p}) \to (q, \dot{p}_i \pi^{ij}, \dot{q}, p_i \pi^{ij} + \dot{p}_k \pi^{kj}{}_{,i} \dot{q}^i).$$

Finally, applying natural involution, we obtain the map $\sim \circ\, TB \circ \alpha^{-1} = T\dot{B}$:

$$T\dot{B}: (q, \dot{q}, p, \dot{p}) \to (q, \dot{q}, \dot{p}_i \pi^{ij}, p_i \pi^{ij} + \dot{p}_k \pi^{kj}{}_{,i} \dot{q}^i).$$

Thus we have $T\dot{B}(dq^i) = \pi^{ij} \partial/\partial \dot{q}^j$, and $T\dot{B}(d\dot{q}^i) = \pi^{ij} \partial/\partial q^j + \pi^{ij}{}_{,k} \dot{q}^k \partial/\partial \dot{q}^j$, which yield the brackets given above in the statement of the theorem. That these brackets obey the Jacobi identity is verified by direct calculation. □

We call this structure the tangent lift to TQ of the Poisson structure on Q.

4.1. $B: T^*Q \to TQ$ is a Poisson morphism

Consider again the bundle map $B: T^*Q \to TQ$ of a Poisson structure on Q. The tangent lift is a Poisson structure on TQ, and there is the canonical symplectic structure, and hence Poisson structure, on T^*Q.

Corollary. B is a Poisson morphism.

Proof. We may ask what Poisson structure on TQ would make B into a Poisson morphism with the canonical structure on T^*Q; this would be determined by the local conditions: $\{q^i, p_j\} = \delta^i{}_j$, and all other brackets zero. In coordinates we have $\dot{q}^i = p_j \pi^{ji}$; so the brackets are given by

$$\begin{aligned}
\{\dot{q}^i, \dot{q}^j\} &= \{\pi^{ir} p_r, \pi^{js} p_s\} \\
&= \pi^{ir}{}_{,k} p_r \{q^k, \pi^{js} p_s\} + \{\pi^{ir} p_r, q^k\} \pi^{js}{}_{,k} p_s \\
&= \pi^{ir}{}_{,k} \pi^{js} p_r \{q^k, p_s\} + \pi^{js}{}_{,k} \pi^{ir} p_s \{q_r, q^k\} \\
&= \pi^{ir}{}_{,k} \pi^{jk} p_r - \pi^{js}{}_{,k} \pi^{ij} p_s \\
&= -\pi^{ji}{}_{,k} \pi^{rk} p_r \\
&= \pi^{ij}{}_{,k} \dot{q}^k
\end{aligned}$$

where we have used the Jacobi identity in the form $\pi^{ir}{}_{,k} \pi^{jk} + \pi^{ji}{}_{,k} \pi^{rk} + \pi^{rj}{}_{,k} \pi^{ik} = 0$, and the skew-symmetry of the matrix of brackets π^{ij}. Now compute

$$\begin{aligned}
\{q^i, \dot{q}^j\} &= -\{q^i, \pi^{jk} p_k\} \\
&= -\pi^{jk} \{q^i, p_k\} \\
&= \pi^{ij}
\end{aligned}$$

The remaining brackets are trivial. Thus we see that B is a Poisson map with the tangent structure on TQ. □

4.2. Further properties of the tangent bracket

We will now see that the tangent Poisson structure is natural with respect to tangents of Poisson maps.

Theorem. Let $f: M_1 \to M_2$ be a Poisson map. Then $Tf: TM_1 \to TM_2$ is a Poisson map for the tangent Poisson structures.

Proof. Let (q^i) and (Q^i) be local charts on M_1 and M_2, respectively, and let $\{q^i, q^j\}_{M_1} = \pi^{ij}$ and $\{Q^i, Q^j\}_{M_2} = \psi^{ij}$ be the brackets on M_1 and M_2, respectively. We will use a comma to denote derivatives on M_1; otherwise we will write out the derivatives explicitly. In addition, I suppress writing composition with f when it is clear from the context where functions live.

Then we have the identities $\dot{Q} = (\partial f/\partial q)\dot{q}$, $Tf(\partial/\partial q^i) = f^j{}_{,i}\,\partial/\partial Q^j$, and $\{Q^i, Q^j\}_{M_2} = \{q^i, q^j\}_{M_1}$.

Thus we have

$$\begin{aligned}\frac{\partial \psi^{ij}}{\partial Q^k}\dot{Q}^k &= \frac{\partial\{Q^i, Q^j\}}{\partial Q^k}\dot{Q}^k \\ &= \frac{\partial\{Q^i, Q^j\}}{\partial Q^k} f^k{}_{,r}\dot{q}^r \\ &= \frac{\partial\{q^i, q^j\}}{\partial q^r}\dot{q}^r \\ &= \pi^{ij}{}_{,r}\dot{q}^r.\end{aligned}$$

Also

$$\begin{aligned}\{Q^i, \dot{Q}^j\}_{TM_2} &= \{Q^i, f^j{}_{,k}\dot{q}^k\} \\ &= \{f^i, f^j{}_{,k}\dot{q}^k\} \\ &= f^i{}_{,r} f^j{}_{,k}\{q^r, \dot{q}^k\} \\ &= f^i{}_{,r} f^j{}_{,k}\pi^{rk} \\ &= \{Q^i, Q^j\}_{M_2} \\ &= \{q^i, q^j\}_{M_1} \\ &= \{q^i, \dot{q}^j\}_{TM_1}.\end{aligned}$$

The remaining brackets are easily seen to be zero. □

5. Momentum maps and the Lie–Poisson bracket

Suppose that an action of G on Q admits an equivariant momentum map $J_Q: T^*Q \to \mathfrak{g}^*$; equivariance means that J intertwines the lifted action of G on T^*Q with the co-adjoint action of G on $\mathfrak{g}^*$, and in particular that J_Q is a Poisson morphism with the symplectic structure on T^*Q and the Lie–Poisson structure on $\mathfrak{g}^*$. Taking tangents, we have a Poisson map $TJ_Q: TT^*Q \to T\mathfrak{g}^* \approx \mathfrak{g}^* \times \mathfrak{g}^*$.

5.1. The Poisson morphism $\mathfrak{g}^* \ltimes \mathfrak{g}^* \to \mathfrak{g}^* \ltimes \mathfrak{g}^*$

In the case described above there is also an action of $G \ltimes \mathfrak{g}$ on TQ, which may be thought of as an action of TG on TQ, and which is given locally by:

$$(g, \xi) \cdot (q, \dot{q}) = (\phi_g(q), T\phi_g\dot{q} + \xi_Q(\phi_g(q)))$$

where ϕ_g denotes the action of the element g on Q and ξ_Q denotes the vector field on Q generated by the infinitesimal action of G on Q. That this is an action of the semidirect product on TQ is verified directly:

$$
\begin{aligned}
(g, \xi)(h, \eta)(q, \dot{q}) &= (g, \xi)(\phi_h(q), T\phi_h(\dot{q}) + \eta_Q(\phi_h(q))) \\
&= (\phi_g(\phi_h(q), T\phi_g T\phi_h(\dot{q}) + T\phi_g \eta_Q(\phi_h(q)) + \xi_Q(\phi_g(\phi_h(q)))) \\
&= (\phi_{gh}(q), T\phi_{gh}(\dot{q}) + T\phi_g \eta_Q(\phi_{g^{-1}}(\phi_{gh}((q))) + \xi_Q(\phi_{gh}(q))) \\
&= (\phi_{gh}(q), T\phi_{gh}(\dot{q}) + (\mathrm{Ad}_g\, \eta)_Q(\phi_{gh}((q))) + \xi_Q(\phi_{gh}(q))) \\
&= (gh, \xi + \mathrm{Ad}_g\, \eta)(q, \dot{q}).
\end{aligned}
$$

Thus the action is seen to be that of the semidirect product $G \circledS \mathfrak{g}$ on TQ. That this calculation is independent of choice of charts is also verified directly. Finally we also observe that this action carries with it an equivariant momentum map, denoted here by J_{TQ}; see Abraham and Marsden (1978). Thus we have an equivariant momentum map $J_{TQ}: T^*TQ \to (\mathfrak{g} \circledS \mathfrak{g})^*$.

Theorem. The following is a commuting diagram of Poisson morphisms:

$$
\begin{array}{ccc}
TT^*Q & \xrightarrow{TJ_Q} & T\mathfrak{g}^* \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \sim} \\
T^*TQ & \xrightarrow[J_{TQ}]{} & (\mathfrak{g} \circledS \mathfrak{g})^*.
\end{array}
$$

We will use the notation $e^{t\dot{q}}(q)$ to denote a curve in Q with the property that at $t=0$ it passes through q with tangent vector $\dot{q}$. Similarly we use $e^{t\xi_Q}(q)$ to denote the flow of the vector field ξ_Q generated by $\xi \in \mathfrak{g}$. Then we have the following lemma.

Lemma.

$$
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \langle (Te^{t\xi_Q}(q))^* \cdot p, \dot{q} \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \langle p, \xi_Q(e^{t\dot{q}}(q)) \rangle.
$$

Proof.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \langle (Te^{t\xi_Q}(q))^* \cdot p, \dot{q} \rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left\langle p, Te^{t\xi_Q} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 e^{s\dot{q}}(q) \right\rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 \langle p, e^{t\xi_Q}(e^{s\dot{q}}(q)) \rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 \left\langle p, \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 e^{t\xi_Q}(e^{s\dot{q}}(q)) \right\rangle \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_0 \langle p, \xi_Q(e^{s\dot{q}}(q)) \rangle.
\end{aligned}
$$

□

Proof of theorem. We first consider the map $TJ_Q : TT^*Q \to g^* \times g^*$:

$$\langle TJ_Q(q, p, \dot{q}, \dot{p}), (\eta, \xi)\rangle = \langle J(p), \eta\rangle + \frac{d}{dt}\bigg|_0 \langle J(e^{t\dot{q}}(q), p + t\dot{p}), \xi\rangle$$

$$= \langle p, \eta_Q\rangle + \frac{d}{dt}\bigg|_0 \langle p + t\dot{p}, \xi_Q(e^{t\dot{q}}(q))\rangle$$

$$= \langle p, \eta_Q\rangle + \langle \dot{p}, \xi_Q(q)\rangle + \frac{d}{dt}\bigg|_0 \langle p, \xi_Q(e^{t\dot{q}}(q))\rangle.$$

Now we consider the map $J_{TQ} : T^*TQ \to g^* \times g^*$:

$$\langle J_{TQ}(q, p, \dot{q}, \dot{p}), (\xi, \eta)\rangle$$

$$= \langle (\dot{p}, p), (\xi, \eta)_{TQ}(q, \dot{q})\rangle$$

$$= \left\langle (\dot{p}, p), \frac{d}{dt}\bigg|_0 (e^{t\dot{q}}(q), Te^{t\xi_Q}\dot{q} + t\eta_Q(e^{t\dot{q}}(q)))\right\rangle$$

$$= \langle \dot{p}, \xi_Q(q)\rangle + \left\langle p, \frac{d}{dt}\bigg|_0 Te^{t\xi_Q}\dot{q}\right\rangle + \left\langle p, \frac{d}{dt}\bigg|_0 t\eta_Q(e^{t\xi_Q}(q))\right\rangle$$

$$= \langle \dot{p}, \xi_Q(q)\rangle + \langle p, \eta_Q\rangle + \left\langle p, \frac{d}{dt}\bigg|_0 Te^{t\xi_Q\dot{q}}\right\rangle$$

comparing this with the previous expression and using the lemma, we see that:

$$\langle TJ_Q(q, p, \dot{q}, \dot{p}), (\eta, \xi)\rangle = \langle J_{TQ}(q, p, \dot{q}, \dot{p}), (\xi, \eta)\rangle. \qquad \square$$

Corollary. The map $g^* \times g^* \to g^* \times g^*$ given by $(\mu, \nu) \to (\nu, \mu)$ is a Poisson map intertwining the tangent Lie–Poisson structure of g^* and the Lie–Poisson structure of $g \circledS g$. In particular, $Tg^* \approx g^* \times g^* \approx (g \circledS g)^*$.

Proof. If either TJ_Q or J_{TQ} are surjective, then this follows directly since all other maps are Poisson maps and the diagram is commutative.

Otherwise, we calculate the respective KKS brackets:

$$g^*: \{\mu^i, \mu^j\} = \pi^{ij}_k \mu^k$$

$$Tg^*: \{\dot{\mu}^i, \dot{\mu}^j\} = \pi^{ij}_k \dot{\mu}^k \qquad \{\mu^i, \dot{\mu}^j\} = \pi^{ij}_k \mu^k \qquad \{\mu^i, \mu^j\} = 0$$

$$(g \circledS g)^*: \{\mu^i, \mu^j\} = \pi^{ij}_k \mu^k \qquad \{\mu^i, \dot{\mu}^j\} = \pi^{ij}_k \dot{\mu}^k \qquad \{\dot{\mu}^i, \dot{\mu}^j\} = 0$$

clearly the map $(\mu, \dot{\mu}) \to (\dot{\mu}, \mu)$ is a Poisson isomorphism. $\square$

5.2. *Natural involution $TTQ \to TTQ$ is a Poisson morphism*

Theorem. If Q is a Poisson manifold, then natural involution $\sim : TTQ \to TTQ$ is a Poisson map for the tangent lift to TTQ of the tangent Poisson structure on TQ.

Similarly, if Q has a closed 2-form, then $\sim : TTQ \to TTQ$ preserves the tangent lift to TTQ of the tangent 2-form on TQ.

Proof. Let (q^i) be a local chart on Q with $\pi^{ij}=\{q^i, q^j\}$ the Poisson brackets on Q. Then we have an induced chart $(q^i, \dot{q}^j)$ on TQ; let us denote these coordinates by x^k so that $x^i=q^i$ and $x^{n+i}=\dot{q}^i$, $1\leq i\leq n$. With this notation, and $1\leq i, j, k\leq n$, we have the following brackets on TQ:

$$\Psi^{ij}=0 \qquad \Psi^{i\,n+j}=\pi^{ij} \qquad \Psi^{n+i\,n+j}=\pi^{ij}{}_{,k}x^{n+k}.$$

Equivalently we have the matrix of brackets:

$$\Psi=\begin{pmatrix} 0 & \pi^{ij} \\ \pi^{ij} & \pi^{ij}{}_{,k}\dot{q}^k \end{pmatrix}.$$

Finally we have the induced chart $(q^i, \dot{q}^j, \delta q^i, \delta\dot{q}^j)$ on TTQ; let us use y^k for these coordinates, so that $y^k=x^k$ if $1\leq k\leq 2n$, $y^{2n+k}=\delta q^k$, and $y^{3n+k}=\delta\dot{q}^k$ if $1\leq k\leq n$. Then for $1\leq i, j, k\leq 2n$, we have

$$\Lambda^{ij}=0 \qquad \Lambda^{i\,2n+j}=\Psi^{ij} \qquad \Lambda^{2n+i\,2n+j}=\Psi^{ij}{}_{,k}y^{2n+k}.$$

Equivalently, the matrix of brackets in the coordinates $(q, \dot{q}, \delta q, \delta\dot{q})$ is

$$\Lambda=\begin{pmatrix} & q & \dot{q} & \delta q & \delta\dot{q} \\ q & 0 & 0 & 0 & \pi^{ij} \\ \dot{q} & 0 & 0 & \pi^{ij} & \pi^{ij}{}_{,k}\dot{q}^k \\ \delta q & 0 & \pi^{ij} & 0 & \pi^{ij}{}_{,k}\delta q^k \\ \delta\dot{q} & \pi^{ij} & \pi^{ij}{}_{,k}\dot{q}^k & \pi^{ij}{}_{,k}\delta q^k & \pi^{ij}{}_{,k,r}\dot{q}^k\delta q^r+\pi^{ij}{}_{,k}\delta\dot{q}^k \end{pmatrix}$$

e.g. $\{\delta\dot{q}^i, \delta q^j\}=\pi^{ij}{}_{,k}\delta q^k$, $\{\delta\dot{q}^i, \delta\dot{q}^j\}=\pi^{ij}{}_{,k,r}\dot{q}^k\delta q^r+\pi^{ij}{}_{,k}\delta\dot{q}^k$, $\{q^i, \delta\dot{q}^j\}=\pi^{ij}$, and so on.

We may now easily see that natural involution sends Λ at $(q, \dot{q}, \delta q, \delta\dot{q})$ to Λ at $(q, \delta q, \dot{q}, \delta\dot{q})$. Therefore natural involution is a Poisson map.

Now consider the case where Q has a 2-form locally given by $\Omega=\Omega_{ij}\,\mathrm{d}q^i\wedge\mathrm{d}q^j$. It will be convenient to express things in terms of matrices; thus in coordinates Ω has the form $\Omega=(\Omega_{ij})$. Then in coordinates $(q, \dot{q})$ on we have the tangent lift of Ω to TQ:

$$\Omega=\begin{pmatrix} \Omega_{ij,k}\dot{q}^k & \Omega_{ij} \\ \Omega_{ij} & 0 \end{pmatrix}.$$

Performing a computation analogous to the Poisson case we get the 2-form Γ on TTQ:

$$\Gamma=\begin{pmatrix} \Omega_{ij,k,r}\dot{q}^k\delta q^r+\Omega_{ij,k}\delta\dot{q}^k & \Omega_{ij,k}\delta q^k & \Omega_{ij,k}\dot{q}^k & \Omega_{ij} \\ \Omega_{ij,k}\delta q^k & 0 & \Omega_{ij} & 0 \\ \Omega_{ij,k}\dot{q}^k & \Omega_{ij} & 0 & 0 \\ \Omega_{ij} & 0 & 0 & 0 \end{pmatrix}$$

as before we see that natural involution sends Γ at $(q, \dot{q}, \delta q, \delta\dot{q})$ to Γ at $(q, \delta q, \dot{q}, \delta\dot{q})$, i.e. natural involution preserves the 2-form Γ. □

6. Tangent Dirac structures

Recall that a Dirac structure on Q is given by a bundle $L\subseteq TQ\oplus T^*Q$ which is maximally isotropic under the natural pairing $\langle\,,\rangle_+$ on $TQ\oplus T^*Q$; in addition, the integrability of L is determined by the vanishing of a 3-tensor on the vector bundle L given by $T_L(e_1\otimes e_2\otimes e_3)=\langle[e_1, e_2], e_3\rangle_+$ where $[e_1, e_2]=([\rho e_1, \rho e_2],\ \rho e_1\lrcorner\,\mathrm{d}\rho^* e_2-\rho e_2\lrcorner\,\mathrm{d}\rho^* e_1-\mathrm{d}\langle e_1, e_2\rangle_-)$, ρ and ρ^* are the natural projections of $TQ\oplus T^*Q$ onto TQ and T^*Q, respectively. See Courant (1990).

Theorem. (i) If $\rho(L)=TQ$ then $\Omega(x,y)=\langle \rho^* \boldsymbol{e}_x, y\rangle$, where $\rho \boldsymbol{e}_x = x$ and $\boldsymbol{e}_x \in \Gamma(L)$, is a well defined differentiable 2-form on Q. (ii) If $\rho^*(L)=T^*Q$, then $\langle \pi_Q | \mathrm{d}f \wedge \mathrm{d}g\rangle = \langle \mathrm{d}f | \rho \boldsymbol{e}_g\rangle$, where $\rho^* \boldsymbol{e}_g = \mathrm{d}g$ and $\boldsymbol{e}_g \in \Gamma(L)$, is a well defined differentiable bi-vector on Q.

In the first case $T_L=0$ is equivalent to $\mathrm{d}\Omega=0$, and Q becomes a pre-symplectic manifold. In the second $T_L=0$ is equivalent to the Jacobi identity for the bracket $\{f,g\}_Q=\langle \pi_Q|\mathrm{d}f \wedge \mathrm{d}g\rangle$, and Q becomes a Poisson manifold.

Proof. These are basic results about Dirac structures; Courant (1989). □

For the sake of a self-contained exposition, we include some more facts about Dirac structures.

Lemma. If $\boldsymbol{a}, \boldsymbol{b}: \mathbb{R}^n \to \mathbb{R}^n$ are linear maps that satisfy

$$\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a}=0 \tag{1}$$

$$\ker \boldsymbol{a} \cap \ker \boldsymbol{b}=\{0\} \tag{2}$$

then $\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{a}-\boldsymbol{b}$ are invertible.

Proof. Suppose that $(\boldsymbol{b}-\boldsymbol{b})x=0$, so that $\boldsymbol{a}x=\boldsymbol{b}x$. Then by (1) we have:

$$\begin{aligned}
&\langle \boldsymbol{a}^*\boldsymbol{b}x|x\rangle+\langle \boldsymbol{b}^*\boldsymbol{a}x|x\rangle=0 \qquad \text{for all } x\in\mathbb{R}^n\\
&\qquad\Rightarrow \langle \boldsymbol{b}x|\boldsymbol{a}x\rangle+\langle \boldsymbol{a}x|\boldsymbol{b}x\rangle=0\\
&\qquad\Rightarrow \langle \boldsymbol{a}x|\boldsymbol{a}x\rangle+\langle \boldsymbol{b}x|\boldsymbol{b}x\rangle=0\\
&\qquad\Rightarrow \|\boldsymbol{a}x\|^2+\|\boldsymbol{b}x\|^2=0
\end{aligned}$$

so that $\boldsymbol{a}x=0$ and $\boldsymbol{b}x=0$, i.e. $x\in\ker \boldsymbol{a}\cap\ker \boldsymbol{b}$, so $x=0$, and $\boldsymbol{a}-\boldsymbol{b}$ is invertible. Similarly for $\boldsymbol{a}+\boldsymbol{b}$. □

Lemma. A Dirac structure at a point q is determined by a pair of maps $\boldsymbol{a}:\mathbb{R}^n\to T_qQ$, $\boldsymbol{b}:\mathbb{R}^n\to T_q^*Q$ such that

$$\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a}=0 \tag{3}$$

$$\ker \boldsymbol{a}\cap\ker \boldsymbol{b}=\{0\}. \tag{4}$$

Proof. Condition (3) is the isotropy of the Dirac structure at q, and condition (4) is the maximallity of the isotropy. □

Corollary. A Dirac structure is determined in a neighbourhood U by a local trivialization $L|_U\approx U\times\mathbb{R}^n$ and a pair of maps $\boldsymbol{a}:U\times\mathbb{R}^n\to T_UQ$, $\boldsymbol{b}:U\times\mathbb{R}^n\to T_U^*Q$, such that

$$\boldsymbol{a}(q):\mathbb{R}^n\to T_qQ \qquad \text{and} \qquad \boldsymbol{b}(q):\mathbb{R}^n\to T_q^*Q \tag{5}$$

$$\boldsymbol{a}^*\boldsymbol{b}+\boldsymbol{b}^*\boldsymbol{a}=0 \tag{6}$$

$$\ker \boldsymbol{a}\cap\ker \boldsymbol{b}=\{0\}. \tag{7}$$

In coordinates (q^i) on U we have $\boldsymbol{a}e_i=a^j{}_i\,\partial/\partial q^j$, $\boldsymbol{b}e_i=b_{ij}\,\mathrm{d}q^j$, where $a^j{}_i, b_{ij}\in C^\infty(U)$. Then the condition $T_L=0$ is equivalent to

$$a^r{}_ib_{js,r}a^s{}_k+a^r{}_jb_{ks,r}a^s{}_i+a^r{}_kb_{is,r}a^s{}_j+a^r{}_ia^s{}_{j,r}b_{ks}+a^r{}_ja^s{}_{k,r}b_{is}+a^r{}_ka^s{}_{i,r}b_{js}=0.$$

Note that this may be written

$$\sum_{\substack{\text{cyclic}\\ \text{sums}}}(a^r{}_ib_{js,r}a^s{}_k+a^r{}_ia^s{}_{j,r}b_{ks})=0.$$

Proof. The first statements follow from the preceding lemmas. The calculation of the components of T_L is straightforward. □

Note that $\boldsymbol{a}: L \to TQ$ and $\boldsymbol{b}: L \to T^*Q$ are really globally defined and are nothing more than the maps ρ and ρ^*. The components $a^j{}_i$ and b_{ij} result from choosing a local trivialization of L over a chart. This is analogous to having global tensors defined on a manifold (e.g. Ω or π_Q) and calculating their components locally.

Consider the maps $\boldsymbol{a}: U \times \mathbb{R}^n \to TQ$ and $\boldsymbol{b}: U \times \mathbb{R}^n \to T^*Q$. We take their tangents to get

$$T\boldsymbol{a}: TU \times T\mathbb{R}^n \to TTQ \qquad \text{and} \qquad T\boldsymbol{b}: TU \times T\mathbb{R}^n \to TT^*Q.$$

Now $TU \times T\mathbb{R}^n \approx TU \times (\mathbb{R}^n \times \mathbb{R}^n)$, and we have coordinates $(q^i, \dot{q}^j, \delta q^i, \delta\dot{q}^j)$ and $(q^i, p_j, \dot{q}^i, \dot{p}_j)$ on TTQ and TT^*Q, respectively, induced by the coordinates (q^i) on U. We use $\boldsymbol{e}_i$ to denote a vector in $\mathbb{R}^n$, and e^i to denote the coordinate functions on $\mathbb{R}^n$ (these are really the $\boldsymbol{e}_i$ with their indices raised by the natural metric on $\mathbb{R}^n$). With these conventions, we now compute $T\boldsymbol{a}$ and $T\boldsymbol{b}$:

$$T\boldsymbol{a}(\dot{q}) = T\boldsymbol{a}\left(\dot{q}^i \frac{\partial}{\partial q^i}\right) = \dot{q}^i \left\{\frac{\partial q^j}{\partial a^i}\frac{\partial}{\partial q^j} + \frac{\partial a^j{}_k e^k}{\partial q^i}\frac{\partial}{\partial \dot{q}^j}\right\}$$

$$= \dot{q}^i \frac{\partial}{\partial q^i} + \dot{q}^i a^j{}_{k,i} e^k \frac{\partial}{\partial \dot{q}^j}$$

$$T\boldsymbol{a}(\dot{e}) = T\boldsymbol{a}\left(\dot{e}^i \frac{\partial}{\partial e^i}\right) = \dot{e}^i \left(\frac{\partial q^j}{\partial e^i}\frac{\partial}{\partial q^j} + \frac{\partial a^j{}_k e^k}{\partial e^i}\frac{\partial}{\partial \dot{q}^k}\right)$$

$$= a^j{}_i \dot{e}^i \frac{\partial}{\partial \dot{q}^j}$$

so we have

$$T\boldsymbol{a}: (q, \boldsymbol{e}, \dot{q}, \dot{\boldsymbol{e}}) \to (q, \boldsymbol{ae}, \dot{q}, \boldsymbol{a}\dot{\boldsymbol{e}} + (\dot{q}\nabla)\boldsymbol{ae}).$$

A similar computation shows that

$$T\boldsymbol{b}: (q, \boldsymbol{e}, \dot{q}, \dot{\boldsymbol{e}}) \to (q, \boldsymbol{be}, \dot{q}, \boldsymbol{b}\dot{\boldsymbol{e}} + (\dot{q}\nabla)\boldsymbol{be})$$

with values in the bundle TT^*Q.

Finally we apply natural and canonical involution to TTQ and TT^*Q respectively, to get bundle maps over $\mathbf{1}_{TQ}$, which we write as $\dot{T}\boldsymbol{a}: TU \times (\mathbb{R}^n \times \mathbb{R}^n) \to TTQ$ and $\dot{T}\boldsymbol{b}: TU \times (\mathbb{R}^n \times \mathbb{R}^n) \to TT^*Q$; locally these maps are given by

$$\dot{T}\boldsymbol{a}|_{(q,\dot{q})}: (\boldsymbol{e}, \dot{\boldsymbol{e}}) \to (\boldsymbol{ae}, \boldsymbol{a}\dot{\boldsymbol{e}} + (\dot{q}\nabla)\boldsymbol{ae})$$

$$\dot{T}\boldsymbol{b}|_{(q,\dot{q})}: (\boldsymbol{e}, \dot{\boldsymbol{e}}) \to (\boldsymbol{b}\dot{\boldsymbol{e}} + (\dot{q}\nabla)\boldsymbol{be}, \boldsymbol{be}).$$

Theorem. The pair of maps $\dot{T}\boldsymbol{a}$ and $\dot{T}\boldsymbol{b}$ determine a Dirac structure on TQ, which we call the tangent lift of the Dirac structure on Q determined by $\boldsymbol{a}$ and $\boldsymbol{b}$.

This construction generalizes the tangent lifts of Poisson and pre-symplectic structures to the tangent bundle.

Proof. We must verify that $\dot{T}\boldsymbol{a}$ and $\dot{T}\boldsymbol{b}$ satisfy the properties $\ker \dot{T}\boldsymbol{a} \cap \ker \dot{T}\boldsymbol{b} = \{0\}$, and $(\dot{T}\boldsymbol{a})^*(\dot{T}\boldsymbol{b}) + (\dot{T}\boldsymbol{b})^*(\dot{T}\boldsymbol{a}) = 0$, as well as the integrability condition.

Suppose that $(\boldsymbol{e}, \dot{\boldsymbol{e}}) \in \ker \dot{T}\boldsymbol{a} \cap \ker \dot{T}\boldsymbol{b}$, so that $\boldsymbol{be} = 0$, $\boldsymbol{ae} = 0$, $\boldsymbol{a}\dot{\boldsymbol{e}} + (\dot{q}\nabla)\boldsymbol{ae} = 0$, and $\boldsymbol{b}\dot{\boldsymbol{e}} + (\dot{q}\nabla)\boldsymbol{be} = 0$. The first pair of equations tells us that $\boldsymbol{e} = 0$, since $\boldsymbol{e} \in \ker \boldsymbol{a} \cap \ker \boldsymbol{b}$, and it follows that $\boldsymbol{a}\dot{\boldsymbol{e}} = 0$ and $\boldsymbol{b}\dot{\boldsymbol{e}} = 0$, since $\boldsymbol{e}$ and $\dot{\boldsymbol{e}}$ are constant. Therefore $\dot{\boldsymbol{e}} = 0$, and we have $\ker \dot{T}\boldsymbol{a} \cap \ker \dot{T}\boldsymbol{b} = \{0\}$.

We now check the skew symmetry of $(\dot{T}\boldsymbol{a})^*(\dot{T}\boldsymbol{b})$; we do it component by component, keeping in mind that $\boldsymbol{a}^*\boldsymbol{b}$ is skew-symmetric, i.e. that $a^k{}_i b_{kj}$ is skew-symmetric with respect to the indices i and j:

$$\langle \dot{T}b(e_i)|\dot{T}a(\dot{e}_j)\rangle = \left\langle \dot{q}^k b_{ri,k}\,\mathrm{d}q^r + b_{ri}\,\mathrm{d}\dot{q}^r \middle| a^k{}_j \frac{\partial}{\partial \dot{q}^k}\right\rangle = b_{ki}a^k{}_j$$

$$\langle \dot{T}b(\dot{e}_j)|\dot{T}a(e_i)\rangle = \left\langle b_{kj}\,\mathrm{d}q^k \middle| a^r{}_i \frac{\partial}{\partial q^r} + \dot{q}^s a^r{}_{i,s}\frac{\partial}{\partial \dot{q}^r}\right\rangle = b_{kj}a^k{}_i$$

$$\begin{aligned}\langle \dot{T}b(e_i)|\dot{T}a(e_j)\rangle &= \left\langle \dot{q}^r b_{ki,r}\,\mathrm{d}q^k + b_{ki}\,\mathrm{d}\dot{q}^k \middle| a^k{}_j \frac{\partial}{\partial q^k} + \dot{q}^k a^r{}_{j,k}\frac{\partial}{\partial \dot{q}^r}\right\rangle \\ &= \dot{q}^r b_{ki,r}a^k{}_j + \dot{q}^r a^k{}_{j,r} b_{ki} \\ &= \dot{q}^r (b_{ki}a^k{}_j)_{,r}\end{aligned}$$

$$\langle \dot{T}b(\dot{e}_i)|\dot{T}a(\dot{e}_j)\rangle = \left\langle b_{ki}\,\mathrm{d}q^k \middle| a^j{}_k \frac{\partial}{\partial \dot{q}^k}\right\rangle = 0.$$

Thus we see that $(\dot{T}a)^*(\dot{T}b)$ is skew-symmetric. It remains only to show that the integrability condition is satisfied. This we do by components after establishing some notation. We will indicate the values over which indices are to be summed. Finally we will use $A^i{}_j$ and B_{ij} for the components of $\dot{T}a$ and $\dot{T}b$, respectively, and if an index greater than n appears with $a^i{}_j$ or b_{ij}, or on q or $\dot{q}$, it will be considered modulo n; thus summations involving only $a^i{}_j$ and b_{ij} or coordinate functions will be over integers $1, 2, \ldots, n$; except in these cases, if no mention is made, summation should be over all integers $1, 2, \ldots, 2n$.

Reading off the equations for these maps, we see the following relations:

$$\begin{aligned} &A^i{}_j = a^i{}_j \text{ and } B_{ij} = \dot{q}^k b_{ij,k} && \text{for } 1 \leqslant i, j \leqslant n \\ &A^i{}_j = \dot{q}^k a^i{}_{j,k} \text{ and } B_{ij} = b_{ij} && \text{for } 1 \leqslant i \leqslant n,\ n+1 \leqslant j \leqslant 2n \\ &A^i{}_j = 0 \text{ and } B_{ij} = b_{ij} && \text{for } 1 \leqslant j \leqslant n,\ n+1 \leqslant i \leqslant 2n \\ &A^i{}_j = a^i{}_j \text{ and } B_{ij} = 0 && \text{for } n+1 \leqslant i, j \leqslant 2n. \end{aligned}$$

We wish to verify that $T_{ijk} = 0$, where

$$T_{ijk} = \sum_{\substack{\text{cyclic}\\ \text{sums}}} (A^r{}_i B_{js,r} A^s{}_k + A^r{}_k A^s{}_{j,r} B_{ks}) \qquad 1 \leqslant i, j, k \leqslant 2n.$$

Suppose that $1 \leqslant i, j, k \leqslant n$, then we have

$$\begin{aligned} A^i{}_r B_{js,r} A^k{}_s &= a^i{}_r \frac{\partial B_{js}}{\partial q} A^k{}_s + \dot{q}^t a^i{}_{r,t} \frac{\partial B_{js}}{\partial \dot{q}^r} A^k{}_s \\ &= \sum_{1 \leqslant s \leqslant n} a^i{}_r \frac{\partial B_{js}}{\partial q^r} a^k{}_s + \sum_{n+1 \leqslant s \leqslant 2n} a^i{}_r \frac{\partial B_{js}}{\partial q^r} (\dot{q}^t a^k{}_{s,t}) \\ &\quad + \dot{q}^t a^r{}_{i,t} \sum_{1 \leqslant s \leqslant n} \frac{\partial B_{js}}{\partial \dot{q}^r} a^k{}_s + \dot{q}^t a^r{}_{i,t} \sum_{n+1 \leqslant s \leqslant 2n} \frac{\partial B_{js}}{\partial \dot{q}^r} \dot{q}^t a^k{}_{s,t} \\ &= \sum_{1 \leqslant s \leqslant n} a^i{}_r \frac{\partial}{\partial q^r} (\dot{q}^t b_{js,t}) a^k{}_s + a^i{}_r \frac{\partial b_{js}}{\partial q} (\dot{q}^t a^s{}_{k,t}) + \dot{q}^t a^{r}{}_{i,t} \{ b_{js,r} a^s{}_k + 0 \} \\ &= a^r{}_i \dot{q}^t b_{js,r} a^s{}_k + a^r{}_i b_{js,r} \dot{q}^t a^s{}_{k,t} + \dot{q}^t a^t{}_{i,t} b_{js,r} a^s{}_k \\ &= \dot{q}^t (a^r{}_i b_{js,r})_{,t} a^s{}_k + a^r{}_i b_{js,r} \dot{q}^t a^s{}_{k,t} \\ &= \dot{q}^t (a^r{}_i b_{js,r} a^s{}_k)_{,t}. \end{aligned}$$

We also have

$$A^r{}_iA^s{}_{j,r}B_{ks} = a^r{}_i\frac{\partial A^s{}_j}{\partial q^r}B_{ks} + \dot{q}^t a^r{}_{i,t}\frac{\partial A^s{}_j}{\partial \dot{q}^r}B_{ks}$$

$$= \sum_{1\leqslant s\leqslant n} a^r{}_i\frac{\partial A^s{}_j}{\partial q^r}\dot{q}^t b_{ks,t} + \sum_{n+1\leqslant 2\leqslant 2n} a^r{}_i\frac{\partial A^s{}_j}{\partial q^r}b_{ks}$$

$$+ \dot{q}^t a^r{}_{i,t}\sum_{1\leqslant s\leqslant n}\frac{\partial A^s{}_j}{\partial \dot{q}^r}\dot{q}^m b_{ks,m} + \dot{q}^t a^r{}_{i,t}\sum_{n+1\leqslant s\leqslant 2n}\frac{\partial A^s{}_j}{\partial \dot{q}^r}b_{ks}$$

$$= \dot{q}^t a^r{}_i a^s{}_{j,r}b_{ks,t} + \dot{q}^t a^r{}_i a^s{}_{j,r,t}b_{ks} + \dot{a}^t a^r{}_{i,t}\{0 + a^s{}_{j,r}b_{ks}\}$$

$$= \dot{q}^t\{a^r{}_i a^s{}_{j,r}b_{ks,t} + a^r{}_i a^s{}_{j,r,t} + b_{ks} + a^r{}_{i,t}a^s{}_{j,r}b_{ks}\}$$

$$= \dot{q}^t(a^r{}_i a^s{}_{j,r}b_{ks})_{,t}$$

so we have $T_{ijk} = 0$ if $1 \leqslant i, j, k \leqslant n$.

Now suppose that $n+1 \leqslant i, j, k \leqslant 2n$. Then we have

$$A^r{}_iB_{js,r}A^s{}_k = \sum_{n+1\leqslant r\leqslant 2n} a^r{}_iB_{js,r}A^s{}_k$$

$$= \sum_{n+1\leqslant r,s\leqslant 2n} a^r{}_iB_{js,r}A^s{}_k$$

$$= 0.$$

We also have

$$A^r{}_iA^s{}_{j,r}B_{ks} = \sum_{n+1\leqslant r,s\leqslant 2n} a^r{}_ia^s{}_{j,r}B_{ks}$$

$$= 0$$

thus $T_{ijk} = 0$ if $n+1 \leqslant i, j, k \leqslant 2n$.

Now suppose that $1 \leqslant i \leqslant n$ and $n+1 \leqslant j, k \leqslant 2n$. Then we have

$$A^r{}_iB_{js,r}A^s{}_k + A^r{}_jB_{ks,r}A^s{}_i + A^r{}_kB_{is,r}A^s{}_k$$

$$= \sum_{1\leqslant r\leqslant n} a^r{}_iB_{js,r}A^s{}_k + \sum_{n+1\leqslant r\leqslant 2n}\dot{q}^t a^r{}_{i,t}\frac{\partial B_{js}}{\partial \dot{q}^r}A^s{}_k$$

$$+ \sum_{n+1\leqslant r\leqslant 2n} a^r{}_j\frac{\partial B_{ks}}{\partial \dot{q}^r}A^s{}_i + \sum_{n+1\leqslant r\leqslant 2n} a^r{}_k\frac{\partial B_{is}}{\partial \dot{q}^r}A^s{}_k$$

$$= \sum_{\substack{1\leqslant r\leqslant n\\ n+1\leqslant s\leqslant 2n}} a^r{}_i\frac{\partial B_{js}}{\partial \dot{q}^r}a^s{}_k + \sum_{1\leqslant r,s\leqslant n} a^r{}_j\frac{\partial B_{ks}}{\partial \dot{q}^r}a^s{}_k$$

$$+ \sum_{\substack{1\leqslant r\leqslant n\\ n+1\leqslant s\leqslant 2n}} a^r{}_j\frac{\partial B_{ks}}{\partial \dot{q}^r}\dot{q}^t a^s{}_{k,t} + \sum_{n+1\leqslant r,s\leqslant 2n} a^r{}_k\frac{\partial B_{is}}{\partial \dot{q}^r}a^s{}_k$$

$$= 0 + 0 + 0 + 0$$

the three remaining terms in our expression are

$$A^r{}_i A^j{}_{s,r} B_{ks} + A^r{}_j A^s{}_{k,r} B_{is} + A^r{}_k A^s{}_{i,r} B_{js}$$

$$= \sum_{1 \leq r,s \leq n} a^r{}_i \frac{\partial A^s{}_j}{\partial q^r} b_{ks} + \sum_{\substack{1 \leq s \leq n \\ n+1 \leq r \leq 2n}} \dot{q}^t a^r{}_{i,t} \frac{\partial A^s{}_j}{\partial \dot{q}^r} b_{ks}$$

$$+ \sum_{n+1 \leq r \leq 2n} a^r{}_j \frac{\partial A^s{}_k}{\partial \dot{q}^r} B_{js} + \sum_{n+1 \leq r \leq 2n} a^r{}_k \frac{\partial A^s{}_i}{\partial \dot{q}^r}$$

$$= 0$$

thus we have $T_{ijk} = 0$ if $1 \leq i \leq n$ and $n+1 \leq j, k \leq 2n$.

We now consider the last case, $1 \leq i, j \leq n$ and $n+1 \leq k \leq 2n$. We have

$$A^r{}_i B_{js,r} A^s{}_k + A^r{}_j B_{ks,r} A^s{}_i + A^r{}_k B_{is,r} A^s{}_j$$

$$= \sum_{1 \leq r \leq n} a^r{}_i \frac{\partial B_{js}}{\partial q^r} A^s{}_k + \sum_{n+1 \leq r \leq 2n} \dot{q}^t a^r{}_{i,t} \frac{\partial B_{js}}{\partial \dot{q}^r} A^s{}_k + \sum_{1 \leq r \leq n} a^r{}_j \frac{\partial B_{ks}}{\partial q^r} A^s{}_i$$

$$+ \sum_{n+1 \leq r \leq 2n} \dot{q}^t a^r{}_{j,t} \frac{\partial B_{js}}{\partial \dot{q}^r} A^s{}_k + \sum_{n+1 \leq r \leq 2n} a^r{}_k \frac{\partial B_{is}}{\partial \dot{q}^r} A^s{}_j$$

$$= a^r{}_i b_{js,r} a^s{}_k + \dot{q}^t a^r{}_{i,t}\{0\} + a^r{}_j b_{ks,r} a^s{}_j + \dot{q}^t a^r{}_{j,t}\{0\} + a^r{}_k \frac{\partial b_{is}}{\partial \dot{q}^r} \dot{q}^t a^s{}_{j,t} + a^r{}_k \frac{\partial \dot{q}^t b_{is,t}}{\partial \dot{q}^r} a^s{}_j$$

$$= a^r{}_i b_{js,r} a^s{}_k + a^j{}_r b_{ks,r} a^s{}_j + a^r{}_k b_{is,r} a^s{}_j .$$

Finally we have the remaining terms

$$A^r{}_i B^s{}_{j,r} B_{ks} + A^r{}_j A^s{}_{k,r} B_{is} + A^r{}_k A^s{}_{i,r} B_{js}$$

$$= \sum_{1 \leq r \leq n} a^r{}_i \frac{\partial A^s{}_j}{\partial \dot{q}^r} B_{ks} + \sum_{n+1 \leq r \leq 2n} \dot{q}^t a^r{}_{i,t} \frac{\partial A^s{}_j}{\partial \dot{q}^r} B_{ks} + \sum_{1 \leq r \leq n} a^r{}_j \frac{\partial A^s{}_k}{\partial q^r} B_{is}$$

$$+ \sum_{n+1 \leq r \leq 2n} \dot{q}^t a^r{}_{j,t} \frac{\partial A^s{}_k}{\partial \dot{q}^r} B_{is} + \sum_{n+1 \leq r \leq 2n} a^r{}_k \frac{\partial A^s{}_i}{\partial q^r} B_{js}$$

$$= a^r{}_i a^s{}_{j,r} b_{ks} + \sum_{n+1 \leq s \leq 2n} a^r{}_j \frac{\partial A^s{}_k}{\partial q^r} b_{is} + \sum_{1 \leq s \leq n} a^r{}_j \frac{\partial A^s{}_k}{\partial q^r} \dot{q}^t b_{is,t}$$

$$+ \sum_{1 \leq s \leq n} a^r{}_k \frac{\partial A^s{}_j}{\partial \dot{q}^r} \dot{q}^r b_{js,t} + \sum_{n+1 \leq s \leq 2n} a^r{}_k \frac{\partial A^s{}_j}{\partial \dot{q}^r} b_{js}$$

$$= a^r{}_i a^s{}_{j,r} b_{ks} + a^r{}_j a^s{}_{k,r} b_{is} + a^r{}_k \frac{\partial \dot{q}^t a^s{}_{i,t}}{\partial \dot{q}^r} b_{js}$$

$$= a^r{}_i a^s{}_{j,r} b_{ks} + a^r{}_j a^s{}_{k,r} b_{is} + a^r{}_k a^s{}_{i,r} b_{js}$$

so $T_{ijk} = 0$ if $1 \leq i, j \leq n$ and $n+1 \leq k \leq 2n$.

Since the integrability tensor T is completely skew-symmetric in i, j, k, the calculations above show that T vanishes identically.

That this construction generalizes the tangent lifts of pre-symplectic and Poisson structures follows immediately from the method of construction. □

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